Renormalization procedure for directed self-organized critical models

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(Received 28 December 1995)

Directed models of self-organized criticality are studied in the framework of a real-space renormalization group of a different type. The identification of a suitable phase space in which to define the renormalization transformation and the coupling with the stationarity condition enables us to clarify the nature of the critical state. The renormalization equations are found to have an attractive fixed point, as expected from the self-critical nature of the model. The values of the critical exponents obtained by this procedure are in excellent agreement with exact results. [S1063-651X(96)10007-6]

PACS number(s): 64.60.Ak, 02.50.−r, 05.40.+j

I. INTRODUCTION

Over the past few years much attention has been devoted to the study of sandpile models, a class of systems introduced by Bak, Tang, and Wiesenfeld as a paradigm of self-organized criticality (SOC) [1]. This term refers to the tendency of large dynamical systems to evolve spontaneously to a critical state having spatial and temporal self-similarity. Many examples of systems showing this kind of behavior have been found in different fields ranging from geology [2] to economics [3] to biology [4]. Recently Pietronero *et al.* [5] introduced a renormalization scheme of a type that is able to describe the self-organized critical state of sandpile models. From the point of view of the renormalization group (RG), a SOC system can be viewed as a system in which no fine tuning of a critical parameter (e.g., the temperature in usual phase transitions) is necessary in order to get criticality. The critical state is reached spontaneously and this corresponds to a completely attractive fixed point of the RG transformation. More recently this scheme has been extended and generalized, to the wide class of dynamical systems having a nonequilibrium stationary state with critical properties, under the denomination of dynamically driven renormalization group [6]. In this paper we discuss the application of such a scheme to the class of directed sandpile models. Dhar and Ramaswamy [7] introduced a variant of the original sandpile model [the Bak-Tang-Wiesenfeld (BTW) model] [1] incorporating a preferential direction into the dynamical rules of the BTW model. They found an exact solution to the model and calculated the critical exponents and the two-point correlation function. The introduction of a preferred direction is, as in percolation problems, a relevant perturbation, i.e., a perturbation that changes the critical behavior of the system. Indeed, the isotropic BTW model and its directed variant belong to different universality classes.

The first step of the method is the identification of a suitable cell for the definition of the cell-to-site transformation. The crucial problem is that the cell has to be invariant under RG transformation [8]. For directed models this means that it has to conserve the orientation with respect to the preferential direction. From this point onward the method can be split into two phases. The first is the identification of the parameters that characterize the static and dynamic properties of the critical state. This corresponds to the identification of the proper phase space in which to study the evolution of the dynamics under scale transformations. The second step is the coupling of the renormalization equations to a stationarity condition that characterizes the driving of the system in its steady state. The stationarity condition provides the weight of the geometrical configurations in the stationary state. This condition allows us to obtain the renormalized stationary parameters that drive the system in the asymptotic steady state.

The RG transformation we obtained evolves under iteration to a completely attractive nontrivial fixed point. This reflects the self-criticality of the model and represents a mechanism for the generation of SOC. We are also able to identify the universality class of various models by studying the basin of attraction of the RG fixed point.

Finally, we compute the critical exponents that describe the system. This is accomplished by making direct use of the scale-invariant dynamics, i.e., the fixed-point properties of the system. The values obtained are in excellent agreement with the exact results of Dhar and Ramaswamy [7].

The outline of the paper is the following. In Sec. II we recall the definition of the model and discuss its phenomenology. In Sec. III we discuss the choice of a suitable cell for the cell-to-site transformation. Section IV is devoted to the choice of the parameter space (phase space) in which to study the evolution of the system under RG transformation. In Sec. V we write the renormalization equations and couple them to the stationarity condition. In Sec. VI we compute the critical exponents and we identify the different universality classes. Finally, in Sec. VII we draw the conclusions.

II. DEFINITION OF THE MODEL

In this section we specify the rules of a directed sandpile model [7] defined on a triangular lattice. Each site i is assigned an integer variable $E(i)$ called energy. The dynamics of the model is defined by the following rules.

(i) Each variable $E(i)$ is initialized with a random integer value such that $0 \leq E(i) < E_c$, where E_c is a threshold value we fix equal to 3.

(ii) A site is chosen at random and its energy is increased by one unit.

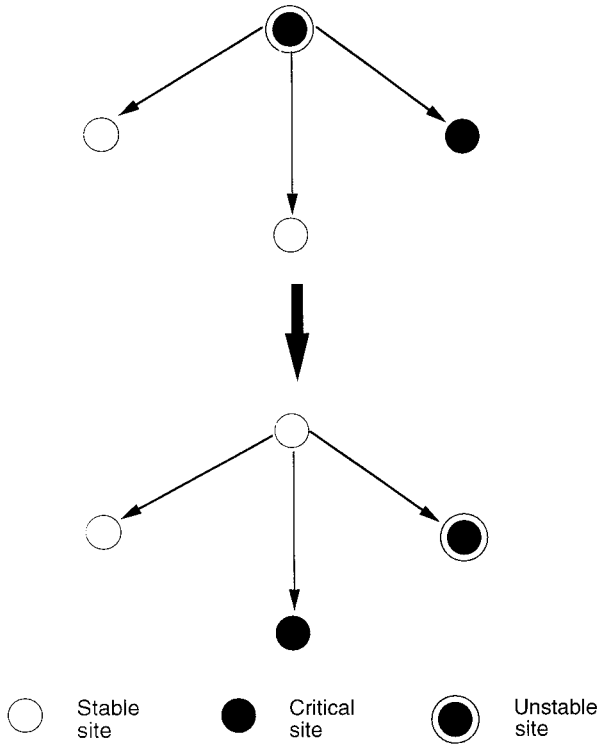

FIG. 1. Definition of the different kinds of sites and the microscopic dynamical rule for the directed sandpile model.

(iii) If $E(i) \geqslant E_c$ for a site i, this site relaxes and transfers its energy to the three nearest-neighbor sites in the half plane defined by the preferential direction (Fig. 1):

$$E(i) \rightarrow E(i) - 3,$$

$$E(j) \rightarrow E(j) + 1 \quad \forall j \in \{3nn\}, \tag{1}$$

where $\{3nn\}$ denotes the set of nearest neighbors sites that receive the energy.

(iv) The relaxation process (avalanche) continues until all the sites have relaxed, i.e., until $E(i) < E_c \forall i$.

(v) Steps (ii) is repeated.

After a transient period the system reaches a stationary critical state in which the energy added to the system equals, on average, the energy flowing out of the system by means of avalanches. This critical state can be characterized by the following set of critical exponents. Denoting with s the number of sites involved in an avalanche, the distribution of avalanches is described by a power law

$$P(s) \sim s^{-\tau}. \tag{2}$$

Analogously,

$$P(r) \sim r^{-\lambda} \tag{3}$$

and

$$P(t) \sim t^{-\alpha}, \tag{4}$$

where r and t are the linear dimension along the preferential direction and the lifetime of the relaxation process, respectively. The variables s, r, and t are linked by the scaling relations

$$t \sim s^{x}, \quad t \sim r^{z}, \quad s \sim r^{D}. \tag{5}$$

We have defined six critical exponents, which are not all independent. By definition we obtain

$$x = \frac{z}{D}. \tag{6}$$

From the identities $P(r)dr = P(s)ds = P(t)dt$, with r, s, and t related by Eq. (5), the following scaling relations can be obtained:

$$\alpha = 1 + \frac{D(\tau - 1)}{z}, \tag{7}$$

$$\lambda = 1 + D(\tau - 1). \tag{8}$$

Therefore, in order to describe the critical behavior of the system, it is enough to compute three critical exponents, for example τ, z, and D.

Dhar and Ramaswamy found that $\tau = \frac{4}{3}$. The value of the dynamical exponent is deduced from the observation that the avalanche front advances at a constant rate of one site at each time step. Therefore

$$r \sim t \tag{9}$$

and we obtain

$$z = 1. \tag{10}$$

Note that in the case of a directed model the exponent D, which links the area of an avalanche (the number of sites involved) and the linear extension of the avalanche in the preferential direction, is not, as in the isotropic case, the fractal dimension of the avalanche clusters D_f, which by virtue of the compactness of the clusters is $D_f = 2$ [7] (see Ref. [9] for the isotropic case). In directed models there are two different lengths characterizing an avalanche: the distance r from the origin of the avalanche to its active front and the avalanche width w. The number of sites involved in an avalanche scales as

$$s \sim rw. \tag{11}$$

Dhar and Ramaswamy showed that the perimeter of the avalanche can be described as two annihilating random walks. This fact yields the scaling for the avalanche width w,

$$w \sim t^{1/2} \sim r^{1/2}. \tag{12}$$

Using this result yields the required scaling relation

$$s \sim r^{3/2}, \tag{13}$$

giving $D = \frac{3}{2}$. The values we obtain in Sec. VI by our RG calculation are in excellent agreement with these results.

The dynamics of the system is characterized by two different time scales [5]: the avalanches are very fast with re-

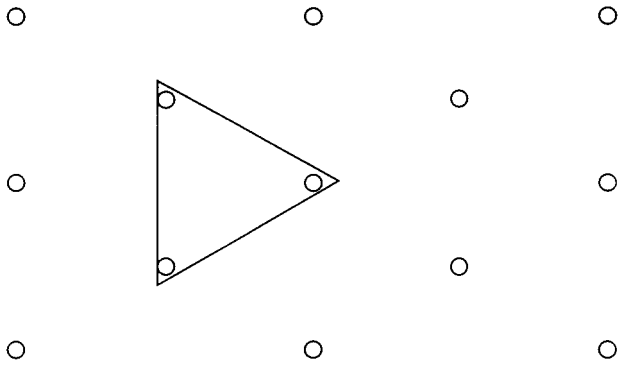

FIG. 2. Elementary cell for a triangular lattice.

spect to the average time between two additions of energy. That means that during the evolution of an avalanche there is no addition of energy. Each avalanche is thus a well defined object. On the other hand, the slow dynamics (the dynamics of additions of energy) accounts for the stationary properties of the system.

III. CHOICE OF A CELL

In order to implement a renormalization-group procedure for this model we use a cell-to-site transformation to average out the degrees of freedom at small scale. First we choose a cell that is invariant under the RG transformation, i.e., whose orientation with respect to the preferential direction does not change under the RG transformation. The natural choice for a cell on the triangular lattice (Fig. 2) is not invariant under the coarse-graining transformation. At each step of the coarse-graining procedure, the cell rotates by 90° with respect to the preferential direction [8]. Therefore the elementary cell is not a good choice because at each step it produces a lattice that has a different orientation with respect to the preferential direction. That means that the dynamical rules could be different at any length scale. Therefore we have to choose a different cell, one that preserves the lattice orientation. The simplest cell satisfying the requirements is shown in Fig. 3 [8]. This cell has a scale factor equal to 2, i.e., the length scales $a^{(k)}$ and $a^{(k+1)}$ at scales (k) and $(k+1)$ respectively, obey $a^{(k+1)}/a^{(k)}=2$.

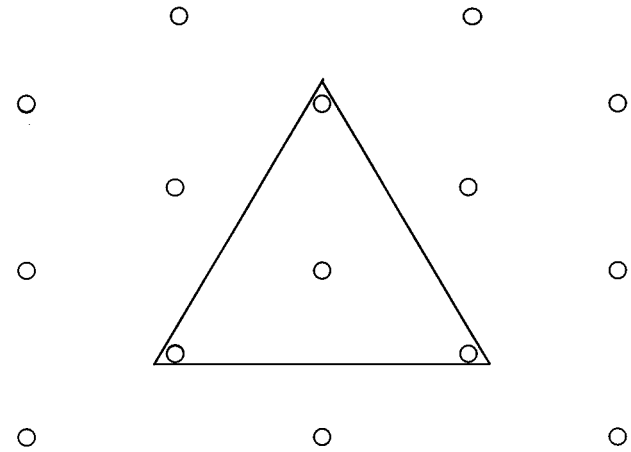

FIG. 3. Four-site cell invariant under the RG transformation used in our RG scheme.

IV. PHASE SPACE

The choice of the phase space in which to define the renormalization transformation has to reflect the property of the system to show a time-scale separation: avalanches are very fast with respect to the average time between two addition of energy.

In full generality we can identify three classes of sites [5] (see Fig. 1).

(i) *Stable sites* are those sites whose energy is far from the critical value. This implies that the addition of a quantum of energy will not induce a relaxation.

(ii) *Critical sites* are those sites whose energy differs by one from the critical value so that the addition of a quantum of energy will induce relaxation

(iii) *Unstable sites* are those sites whose energy equals or exceeds the critical value so that they will relax at the next time step.

Between avalanches the system contains only critical and stable sites so that the critical state can be described by a probability ρ, which is the density of critical sites. The parameter ρ can be viewed as a control parameter that drives the balance between the energy added to the system and the energy flowing out of the system with the avalanches. However, it is not a control parameter in the usual sense of critical systems. Tuning of ρ to get criticality is not necessary because successive applications of the RG transformation adjust it to its critical value.

The above representation of a configuration can be extended to describe the system at any length scale (k). A coarse-grained site is said to be *stable* if the addition of a quantum of energy $\delta E^{(k)}$ at the scale (k) will not induce relaxations towards neighboring sites. The quantity $\delta E^{(k)}$ is also the mean energy that two sites exchange at the length scale (k). On the other hand, a site is called *critical* if the addition of $\delta E^{(k)}$ will induce a relaxation. With these definitions in mind, we introduce the parameter $\rho^{(k)}$ as the density of critical sites at the scale (k).

The dynamic rule itself changes under the coarse-graining transformation: in the microscopic dynamics a site relaxes into three neighbors, but this is not necessarily the case for a coarse-grained cell. In order to include the possible proliferations of parameters occurring under coarse-graining transformation, we define, independently of what happens at the minimal scale, a probability vector $\vec{P}^{(k)}$ whose components represent the probabilities for the possible mechanisms for energy transfer at a coarse-grained scale (k).

Given a series of relaxations of sites at a scale (k), we have to specify the corresponding relaxations of cells at the scale (k). These are the relaxations of sites at the scale $(k+1)$. A site at the scale $(k+1)$ relaxes if the relaxation processes at the lower scale (k) span the cell according to the spanning condition and transfers the energy to some neighboring cells. A cell is said to be spanned if the relaxation processes involve at least two different sites. In our scheme the relaxation at a generic scale (k) can occur in five different ways (Fig. 4): $P_1^{(k)}$, the probability for the energy trans-

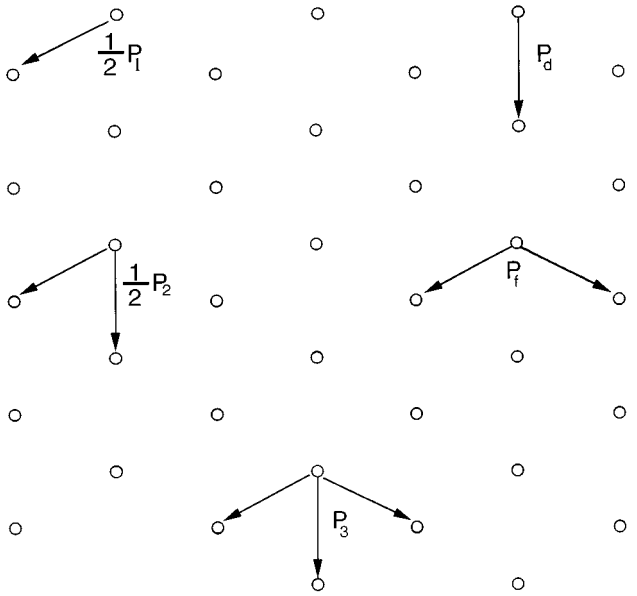

FIG. 4. Five different ways for the energy transfer. Their probabilities correspond to the five components of the vector $\vec{P}$.

fer to one lateral neighboring site; $P_d^{(k)}$ the probability for the energy transfer to the central neighboring site; $P_2^{(k)}$ the probability for the energy transfer to the central and to one of the lateral neighboring sites; $P_f^{(k)}$ the probability for the energy transfer to both the lateral neighboring sites; and $P_3^{(k)}$ the probability for the energy transfer to the three neighboring sites.

We have chosen a limited parameter space that does not allow for relaxation into next nearest neighbors or backward relaxations (and therefore also no multiple relaxations). This choice is obviously an approximation, but one that is always present in real-space renormalization-group calculations. We have incorporated the preferred direction into our scheme through the choice of the parameter space and the spanning condition. A more generalized scheme should differentiate between isotropic and directed models only through the spanning condition. The spanning condition for directed models would be to consider only processes that distribute energy in the preferred direction.

We note that among the processes that contribute to the different components of $\vec{P}^{(k)}$ there are also processes that involve sites 2 and 3 in Fig. 5. Such processes do not transfer energy to the cells to which sites 2 and 3 belong; those processes, in fact, do not span but just touch these cells in the process of energy transfer to other cells. Therefore sites 2 and 3 can only transfer energy out of their own cells at scale (k). In order to take into account such processes we consider an apparent cell composed of six sites, the four sites of the chosen cell plus sites 2 and 3 of Fig. 5. The statistical weights of processes not involving sites 2 and 3 will be unaltered. The cell used for the renormalization procedure is the four-site cell and the introduction of the apparent cell is just a way to take into account processes in which site 1 transfers energy to sites 4, 5, or 6 in an indirect way via the sites 2 or 3.

The model we have defined in Sec. II corresponds, at the minimal scale, to the vector $\vec{P}^{(0)} \equiv (0,0,0,0,1)$. This model is the one defined by Dhar and Ramaswamy on a triangular

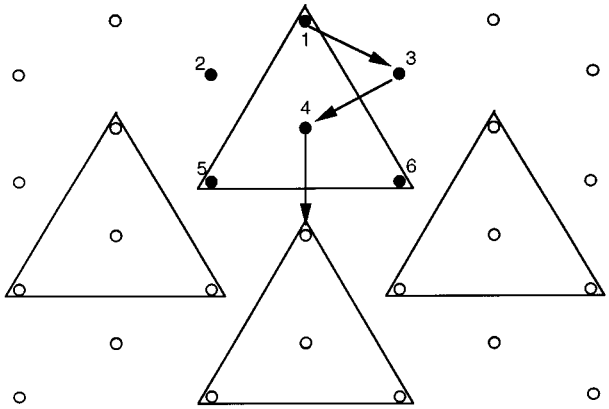

FIG. 5. Six sites (in black) that constitute the so-called apparent cell, i.e., the cell including all the sites involved in the process of energy transfer from a four-site cell. Arrows indicate an energy transfer process that involves a site external to the four-site cell.

lattice [7]. We will consider also another model corresponding to the vector $\vec{P}^{(0)} \equiv (0,0,0,1,0)$. This model is equivalent to the Dhar-Ramaswamy model on a square lattice. In fact, one can obtain our triangular lattice by compressing a square lattice in the diagonal direction and considering the same microscopic rules, i.e., the transfer to the nearest-neighboring sites in the preferential direction on the square lattice becomes the transfer P_f on the triangular lattice.

V. RENORMALIZATION EQUATIONS

In order to write the renormalization transformation it is necessary to consider all the possible processes at the generic scale (k) that contribute to a single process at the scale $(k+1)$. The RG transformation is given by the sum of the statistical weights of the dynamical processes allowed by the spanning condition. A transfer processes satisfies the spanning condition if it starts from the central site or the one at the top of the cell (top site). The probability for a transfer process to start in one of the two sites is proportional to the probability for each site to receive energy by external addition or by transfer from other sites during an avalanche. Because only two sites external to the cell can transfer energy to the central site the probability for a process to begin at the central site is $P_c \sim P_l + P_2 + 2P_f + 2P_3$. In the same way one can find the probability P_t for a process to start at the top site. Normalizing these probabilities with the condition $P_c + P_t = 1$ one obtains

$$P_c = \frac{P_l + P_2 + 2P_f + 2P_3}{2P_l + P_d + 3P_2 + 4P_f + 5P_3},$$

$$P_t = \frac{P_l + P_d + 2P_2 + 2P_f + 3P_3}{2P_l + P_d + 3P_2 + 4P_f + 5P_3}. \tag{14}$$

The statistical weight of a process has to include a factor that represents the probability for obtaining the corresponding configuration. The probability for a critical cell to be in a configuration with α critical sites is

$$W_\alpha(\rho) = \frac{\rho^\alpha (1-\rho)^{6-\alpha}}{N_\alpha \Sigma_{\alpha=2}^{6} \rho^\alpha (1-\rho)^{6-\alpha}}, \tag{15}$$

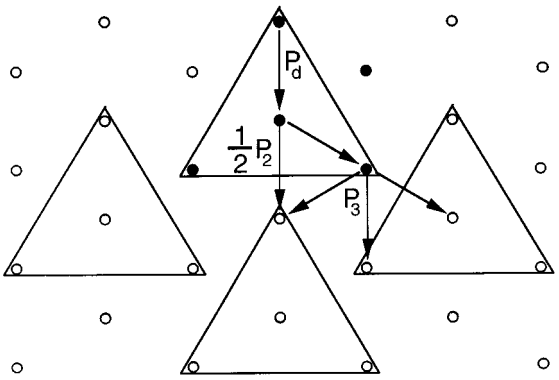

FIG. 6. Example of one energy transfer process with statistical weight equal to $\frac{1}{2}P_dP_2P_3$.

where N_α is the number of configurations with α critical sites. Denoting by ω_α the index ranging over all configurations with α critical sites, where $\omega_\alpha=1,\ldots,N_\alpha$, the probability of a transfer process of kind x will be

$$P_x^{(k+1)}=\sum_\alpha W_\alpha(\rho)\sum_{w_\alpha=1}^{N_\alpha} P_x^{(k+1)}(\omega_\alpha), \tag{16}$$

where $P_x^{(k+1)}(\omega_\alpha)$ is the probability for a transfer of kind x from the cell ω_α at the scale $(k+1)$, with the normalization condition

$$\sum_x \sum_{\omega_\alpha=1}^{N_\alpha} P_x^{(k+1)}(\omega_\alpha)=N_\alpha.$$

The probability $P_x^{(k+1)}(\omega_\alpha)$ is the sum over all the different transfer processes, from the cell ω_α at the scale (k), contributing to the transfer process of kind x at the scale $(k+1)$:

$$P_x^{(k+1)}(\omega_\alpha)=P_t^{(k)}\sum_i Q_{i,t}^{x,(k)}(\omega_\alpha)+P_c^{(k)}\sum_i Q_{i,c}^{x,(k)}(\omega_\alpha), \tag{17}$$

where $Q_{i,j}^{x}$ is the statistical weight of the ith process started at the jth site, which contributes to an x process. The indices in parentheses refer to the length scale, while P_t and P_c are the probabilities that the transfer process starts from the top or the central site, respectively. In Fig. 6 an example is shown of a transfer process that transfers energy to two neighboring cells. Its weight is given by $Q_{i,v}^{2,(k)}=P_d^{(k)}\frac{1}{2}P_2^{(k)}P_3^{(k)}$.

The number of processes that contribute to Eq. (16) is large (several thousands) so that it is impractical to find all the configurations by hand. In order to find all the processes at a certain scale that contribute to a simple process at a higher scale we have proceeded as follows. A site can transfer energy to up to three sites (the nearest neighbors in the preferential direction). The possibility for energy transfer is represented by a binary bond that is one or zero depending on whether or not it participated in the relaxation process. In the apparent cell there are six sites and so we have eighteen different bonds. Each relaxation process then corresponds to a particular sequence of bits. In this way it is possible to generate all the possible relaxation processes. Among all these processes one has to choose the ones that contribute to the renormalization equations. One has to discard all the processes that do not correspond to dynamical processes (for example, sequences that represent a relaxation of a stable site) and to accept just the processes that fulfill the imposed spanning condition. By implementing this algorithm one reproduces all the processes that contribute to the renormalization equations.

Equations (16) and (17) are the renormalization equations for the dynamical variables. We have yet to find how the configurational parameter $\rho^{(k)}$ is renormalized. This equation will express the stationarity of the critical state. It is derived from the following expression, which states the balance between the energy input to a cell and the energy flowing out of it:

$$\delta E^{(k+1)}=\rho^{(k+1)}\delta E^{(k+1)}[P_1^{(k+1)}+P_d^{(k+1)}+2(P_2^{(k+1)}+P_f^{(k+1)})+3P_3^{(k+1)}], \tag{18}$$

where $\delta E^{(k+1)}$ is the quantum of energy added to a cell at the scale $(k+1)$, i.e., the mean energy exchanged between two cells at that scale. From the previous relation we easily obtain

$$\rho^{(k+1)}=\frac{1}{P_l^{(k+1)}+P_d^{(k+1)}+2(P_2^{(k+1)}+P_f^{(k+1)})+3P_3^{(k+1)}}. \tag{19}$$

Equation (19) provides, independently of the definition of $\delta E^{(k+1)}$, the renormalized density of critical sites at the scale $(k+1)$. Moreover, it couples dynamical parameters of the system to the static one, creating a feedback between the control parameter $\rho^{(k)}$ and the dynamics of the system. This is the mechanism generating the self-criticality of the model. The complete RG equations are then

$$\begin{aligned}P_x^{(k+1)}&=\sum_\alpha W_\alpha(\rho)\sum_{\omega_\alpha=1}^{N_\alpha}\left(P_t^{(k)}\sum_i Q_{i,t}^{x,(k)}(\omega_\alpha)+P_c^{(k)}\sum_i Q_{i,c}^{x,(k)}(\omega_\alpha)\right),\\ \rho^{(k+1)}&=[P_l^{(k+1)}+P_d^{(k+1)}+2(P_2^{(k+1)}+P_f^{(k+1)})+3P_3^{(k+1)}]^{-1}.\end{aligned} \tag{20}$$

Starting with an arbitrary initial condition $(\rho^{(0)},\vec{P}^{(0)})$, we can study the flow diagram and the fixed points just be iterating the renormalization equations (20). We found that the system of equations has a stable fixed point $(\rho^*,\vec{P}^*)$. In Table I we show the evolution under scale transformation of the parameters and the numerical values of the fixed point parameters for the two models defined by the minimum scale vectors $\vec{P}^{(0)}\equiv(0,0,0,0,1)$ and $\vec{P}^{(0)}\equiv(0,0,0,1,0)$ for two arbitrary values of $\rho^{(0)}$. Both models evolve asymptotically towards the same stable fixed point, which corresponds to the same scale-invariant dynamics. This means that the two models belong to the same universality class.

In addition, we checked that systems for which $P_f^{(0)}+P_2^{(0)}+P_3^{(0)}\neq 0$ converge towards the same stable fixed point, i.e., they belong to the same universality class. Systems for which $P_f^{(0)}+P_{2(0)}+P_3^{(0)}=0$ converge to some

TABLE I. Iteration of the RG equations for the parameters of two microscopic models [(a) with $P_3^{(0)}=1$ and (b) with $P_f^{(0)}=1$] corresponding to $\vec{P}^{(0)}\equiv(0,0,0,0,1)$ and $\vec{P}^{(0)}\equiv(0,0,0,1,0)$.

(k)	P_l	P_d	P_2	P_f	P_3	ρ
			(a)			
0	0.000	0.000	0.000	0.000	1.000	0.900
1	0.000	0.000	0.003	0.000	0.997	0.334
2	0.000	0.506	0.259	0.000	0.235	0.578
3	0.035	0.342	0.307	0.007	0.308	0.518
∞	0.058	0.308	0.319	0.012	0.303	0.516
			(b)			
0	0.000	0.000	0.000	1.000	0.000	0.400
1	0.144	0.000	0.056	0.028	0.772	0.380
2	0.028	0.370	0.290	0.005	0.306	0.524
3	0.049	0.327	0.305	0.010	0.309	0.517
∞	0.058	0.308	0.319	0.012	0.303	0.516

other unstable fixed points. Let us consider, for example, the system with $P_d^{(0)}=1$. This is a fixed point because the dynamical evolution cannot generate lateral energy transfers. This fixed point is unstable: any deviation from $P_d^{(0)}=1$ drives the system away from it.

VI. CALCULATION OF THE CRITICAL EXPONENTS

As we have seen in the previous sections, the fixed point of our renormalization scheme has a completely attractive nature. This implies that it is not possible to compute the critical exponents by the standard method used in critical phenomena. The calculation of the critical exponents τ and z is carried out using the fixed point dynamics, thus overcoming the problem posed by the complete attractiveness of the RG flow.

In Sec. II we have shown that in order to fully characterize the critical behavior of the system, it is necessary to calculate three independent exponents, e.g., D, the exponent that describes how the area of an avalanche scales with its linear dimension in the preferential direction; the avalanche exponent τ, and the dynamical exponent z.

Let us start with the calculation of D. We have already shown that, at the minimal scale, the two models with $\vec{P}^{(0)}\equiv(0,0,0,0,1)$ and $\vec{P}^{(0)}\equiv(0,0,0,1,0)$ produce avalanches whose area scales with an exponent $D=\frac{3}{2}$. The reasoning that leads to $D=\frac{3}{2}$ in these models can be applied to any system satisfying $P_f^{(0)}+P_2^{(0)}+P_3^{(0)}\neq 0$ at any scale (k). In particular, this holds for the scale-invariant dynamics corresponding to the nontrivial fixed point $(\rho^*,\vec{P}^*)$ (Table I).

As in [5], for the calculation of the avalanche exponent τ we define K as the probability that a relaxation process is limited between the scales (k) and $(k+1)$ and does not extend further:

$$K=\frac{\int_{\alpha^{(k)}}^{\alpha^{(k+1)}}P(r)dr}{\int_{\alpha^{(k)}}^{\infty}P(r)dr}. \tag{21}$$

By simple scaling arguments we get that

$$P(r)\sim r^{D(\tau-1)-1}.$$

Inserting this expression in (21) we obtain

$$K=1-2^{(3/2)(1-\tau)}$$

and

$$\tau=1-\frac{2\ln(1-K)}{3\ln 2}, \tag{22}$$

where we used the following value as the scale factor of our coarse-grained cell:

$$\frac{a^{(k+1)}}{a^{(k)}}=2. \tag{23}$$

K can expressed in terms of the scale-invariant dynamics as the probability that at the scale (k) the energy is transferred to stable cells:

$$K=(P_l^*+P_d^*)(1-\rho^*)+(P_f^*+P_2^*)(1-\rho^*)^2+P_3^*(1-\rho^*)^3. \tag{24}$$

Using the fixed point parameters and inserting expression (24) into (22), we get the value

$$\tau=1.328,$$

in excellent agreement with the exact result $\tau=\frac{4}{3}$ obtained by Dhar and Ramaswamy [7].

The dynamical exponent links the linear extension of an avalanche to the time needed for its evolution. As we already pointed out, for directed models one considers the linear extension in the preferential direction

$$\langle t^{(k)}\rangle\sim a^{(k)^z},$$

where $\langle t^{(k)}\rangle$ is the average time it takes for a dynamical process at the scale (k) to cross the cell. Comparing two consecutive scales, one obtains

$$\frac{\langle t^{(k+1)}\rangle}{\langle t^{(k)}\rangle}=\left(\frac{a^{(k+1)}}{a^{(k)}}\right)^z=2^z, \tag{25}$$

so that

$$z=\frac{\ln\dfrac{\langle t^{(k+1)}\rangle}{\langle t^{(k)}\rangle}}{\ln 2}. \tag{26}$$

The calculation of the dynamical exponent z is performed by the following procedure. A process that transfers energy out of a cell at the length scale $(k+1)$ is composed of many

subprocesses at the scale (k). We denote by $M^x_{i,j}$ the number of steps involved in the ith x coarse-grained process starting at the jth site. Thus, at a scale $(k+1)$ the process characterized by $M^x_{i,j}$ runs in a time given by

$$t^x_{i,j} = M^x_{i,j} \langle t^{(k)} \rangle .$$

The average time at the length scale $(k+1)$ is a weighted average of all the process times $t^x_{i,j}$:

$$\langle t^{(k+1)} \rangle = \langle t^{(k)} \rangle \sum_{\alpha,x} W(\alpha) \sum_{\omega_\alpha} \left(P_t^{(k)} \sum_i Q^{x,(k)}_{i,t}(\omega_\alpha) M^x_{i,t} + P_c^{(k)} \sum_i Q^{x,(k)}_{i,c}(\omega_\alpha) M^x_{i,c} \right). \tag{27}$$

Inserting Eq. (27) into Eq. (26) yields an expression for the dynamical exponent

$$z = \frac{\ln\left[\sum_{\alpha,x} W(\alpha) \sum_{\omega_\alpha} \left(P_t^{(k)} \sum_i Q^{x,(k)}_{i,t}(\omega_\alpha) M^x_{i,t} + P_c^{(k)} \sum_i Q^{x,(k)}_{i,c}(\omega_\alpha) M^x_{i,c} \right) \right]}{\ln 2}. \tag{28}$$

Summing over all contributions gives

$$z = 1.001, \tag{29}$$

which is almost identical to the exact result. In fact, the front of an avalanche moves at a constant rate of one unit per time step in the preferential direction at any scale (k).

We stress that despite the good agreement, the differences between the exact results and the results we obtained for z and τ are not due to rounding errors. Indeed, the renormalization procedure we used to compute the critical exponents is not an exact procedure: we have neglected, for example, the next-nearest-neighbor energy transfer.

VII. CONCLUSION

In this paper we have presented a renormalization scheme of a different type for the study of directed sandpile models. This approach represents an application of the so-called dynamically driven renormalization group and it follows the strategy used in [5] for the nondirected sandpile models. The method consists of two steps: the identification of a suitable parametrization of the static and dynamic properties of the system and the coupling of the renormalization equations to a condition that expresses the stationarity of the critical state. In this way we obtain the RG transformations characterizing the evolution of the system under a change of scale. This method enables us to characterize the nature of the critical state and its scale invariant dynamics.

As in the case of isotropic models [5] the RG transformation was found to have an attractive stable fixed point that accounts for the self-critical character of these models. We identified the universality classes of several models by studying the basin of attraction of the fixed point. In addition, we computed analytically the critical exponents and obtained values that are in excellent agreement with the exact results of Dhar and Ramaswamy [7].

ACKNOWLEDGMENTS

The authors are grateful to L. Pietronero, A. Vespignani, and S. Zapperi for interesting discussions. A.B. wishes to thank O. Biham for his guidance.

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